IN SEARCH OF THE Prototypical Fraction

Vince Wright
Australian Catholic University
<vince.wright@acu.edu.au>

Vince Wright makes a convincing argument for presenting children with a different 'prototype' of a fraction to the typical one-half. Consider how the prototype that Wright mentions may be applied to a variety of fraction concepts. We are sure that you will never look at a doughnut in quite the same way.

Introduction

When you think of a prototype, an image of some innovatively designed communication device, automobile or aeroplane probably appears. It is the first of its kind, the prime example from which all other similar examples are copied. However, in psychology, prototypes are single, typical examples of a concept that embody the key features. Along with rules and trusted solutions, prototypes are important tools for generalisation that help us to see similar properties in different situations (Mancy, 2010). We give names to these examples, using words and symbols, so we can talk about them.

In the aviation or motor industry a huge amount of effort goes into creating prototypes that have the required functionality. A prototypical Concorde that flies faster than the speed of sound but is blown around like a beach ball in strong winds and has uncomfortable passenger seats will not fly (so to speak). In learning, prototypes give you a way to chunk together a whole lot of information and process it as a single artefact. Compression of many features into one example helps you to free memory resources (Bransford, Brown & Cocking, 2000; Fauconnier & Turner, 2008; Gray & Tall, 2007).

While prototypes are powerful thinking tools, compression comes with consequence. Prototypes also restrict the way in which you view situations. For example, students often possess a prototypical triangle, the upward

facing equilateral triangle. This inhibits their ability to classify other three-sided polygons as triangles and to work with other properties of triangles such as angles and symmetry.

Ask your students to write down the first fraction that enters their head and draw some picture of it. I have strong anecdotal evidence that most students record something like this:

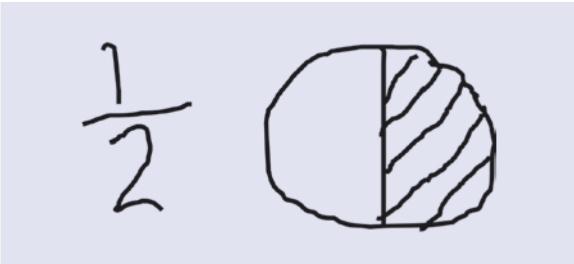

Figure 1. Prototype of one-half.

Ask your students to explain the meaning of (in Figure 1) the symbols one and two and most will say, "One out of two." Gould (2005) reminds us that this part–whole prototype involves two whole number counts which may limit the ability of students to consider one-half as a single quantity. Given license, your students will draw a variety of representations, including lengths, areas, clocks, sets and number lines. The sad fact is that one-half is such a poor prototype that Kath Hart described it as "an honorary whole number" (Hart et al., 1981). The pizza is a perfectly reasonable representation of one-half but it is also an unhelpful prototype. Try using a pizza model to decide which fraction, two-thirds or five-sevenths, is larger (no protractors and calculators allowed). Dividing a circle model into equal pieces accurately enough to solve the problem is difficult.

Multi-functioning prototypes

An advertisement for a new model of car will be very unlikely to boast that the vehicle just moves. Saying one-half is "one out of two" is a bit like that. The television commercial will likely describe how the car accelerates like a hungry cheetah, offers sofa-like comfort and has style that makes other drivers envious. A fraction prototype should have multiple features as well. You owe a lot to previous fraction prototype designers for the modern improvements you now take for granted (Kieren, 1993; Post, Behr, Lesh & Wachsmuth, 1986; Streefland, 1993).

Let us introduce the stunning six-tenths as an alternative prototype and describe its features. It is a multi-functioning fraction with excellent optional features to suit different conditions.

The measurement feature

Measurement is the basic multi-purpose mode since it expresses a fraction as a number. The measurement feature is ideal in situations like creating a number line where one is fixed. Many young fraction drivers, like your students, do not realise the ubiquitous 'whole' for all fractions is actually the number one (Yoshida & Sawano, 2002). Suppose you have a length of paper. The beginning of the strip marks zero and the end of the strip marks one. Where is six-tenths, our prototypical fraction?

First, you need to find one-tenth; then count six of measures of one-tenth. So $\frac{6}{10}$ is short-hand for $\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$. This is a useful addition to the simplistic 'out of' feature since it helps us make sense of improper fractions like thirteen-tenths.

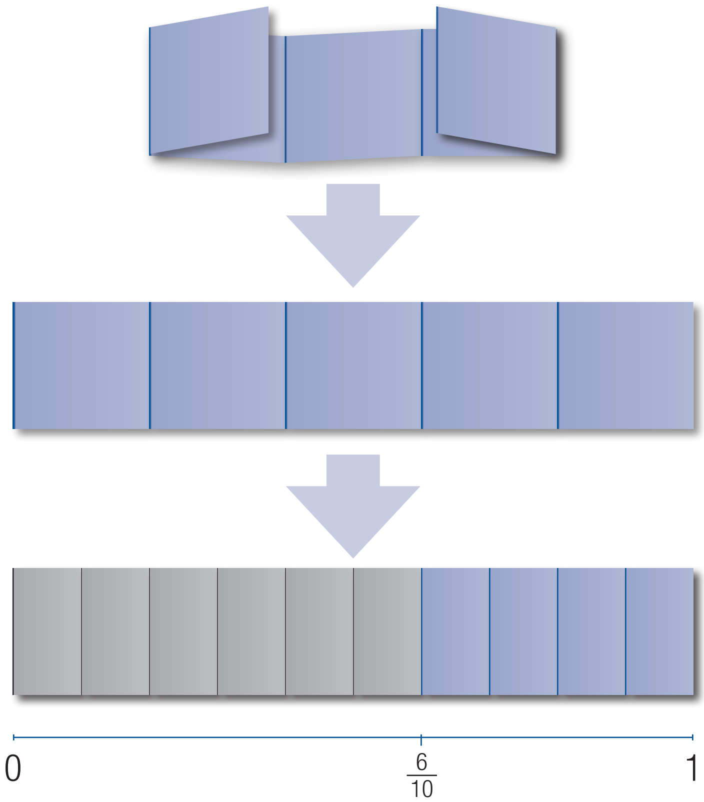

Figure 2. Paper strip model for six-tenths

Finding one-fifth of one-half produces the same equally-sized piece as finding one-half of one-fifth. You could write this as $\frac{1}{2} \times \frac{1}{5} = \frac{1}{5} \times \frac{1}{2}$ which looks familiar as it involves the same commutative property of multiplication as $4 \times 3 = 3 \times 4$. It is nice when the features of a prototype for one object, like whole number, mirror those of different objects, like fractional numbers. There is less new stuff for you to learn. Another interesting thing is that you get the same result only if you measure the same attribute (feature). If you fold the strip attending to area, not length, you can get this model of six-tenths (Figure 3). Different models are useful for different purposes. Your prototype needs to be versatile enough to meet these demands.

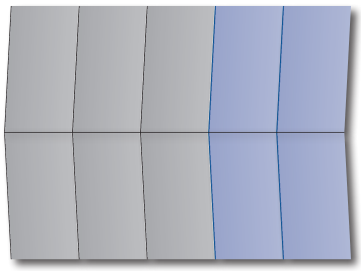

Figure 3. Area model of six-tenths.

The measure feature also allows us to change the appearance of six-tenths while retaining all of its other characteristics, especially its size. So you can rename six-tenths to suit your mood. Firstly, there are literally an infinite number of fraction names for it. For example, if you combine tenths in lots of two you get three-fifths. If you split tenths in half, six-tenths becomes twelve-twentieths.

If a fraction does not spin your wheels (pun intended) then you can change your prototype's appearance to a percentage (60%) or a decimal (0.6). These appearances are very handy for shopping sprees and ordering materials that are sized in metres, kilograms, litres and other measurement units. But do not let appearances be deceiving: you will still have your six-tenths, since sixty-hundredths (60%) is an equivalent fraction and "point six" just means six-tenths.

The operator feature

A fraction prototype, like a car, may look pretty but can it handle interactions with other numbers? Ferraris may be great sports cars but try using one to jump start another vehicle, let alone tow a caravan or trailer, and Hummers can be impressive except in crowded traffic. Fractions act on other numbers and six-tenths is no exception. Suppose you want to find six-tenths of \$40. You know from the measurement feature that six-tenths is 'six lots of one-tenth'. You also know that multiplication is the 'of' function, like 4×7 means "four sets of seven." So you can find one tenth and multiply that by six.

Sometimes it is easy to forget that your prototype will interact with other numbers in the same way, no matter what appearance it has. Six-tenths of \$40 has the same answer as 60% of \$40 and 0.6×40. As an operator, your prototypical fraction obeys the same rules as more primitive prototypes like whole numbers. If it is okay to work out 'ninety sets of six' as 'six sets of ninety', it must be okay to find 'ninety lots of six-tenths' as 'six-tenths of ninety'. The digital display of your

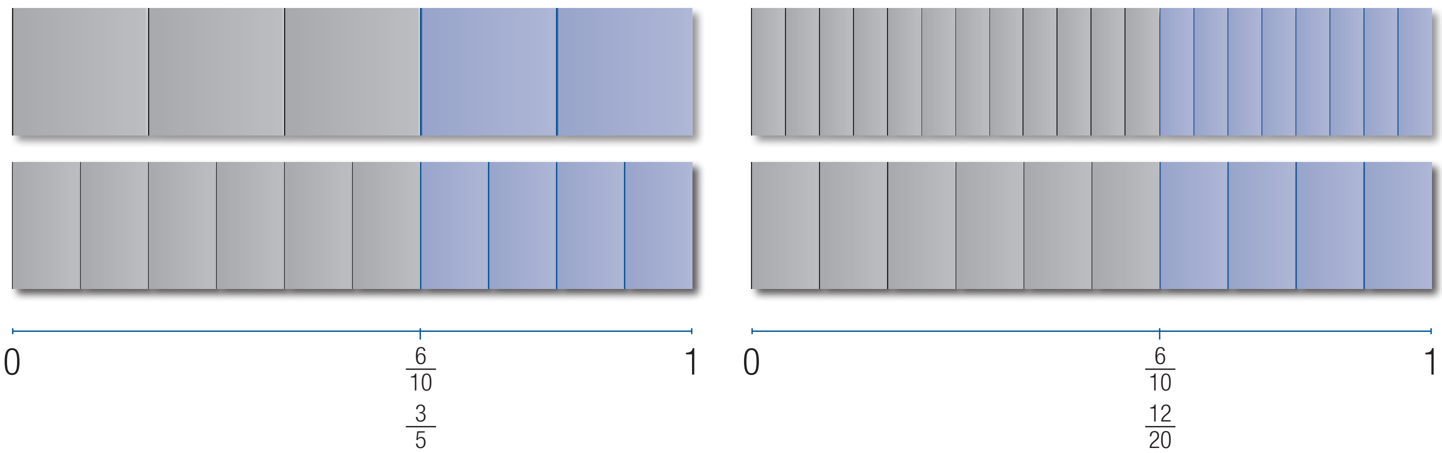

Figure 4. Equivalent measures to six-tenths.

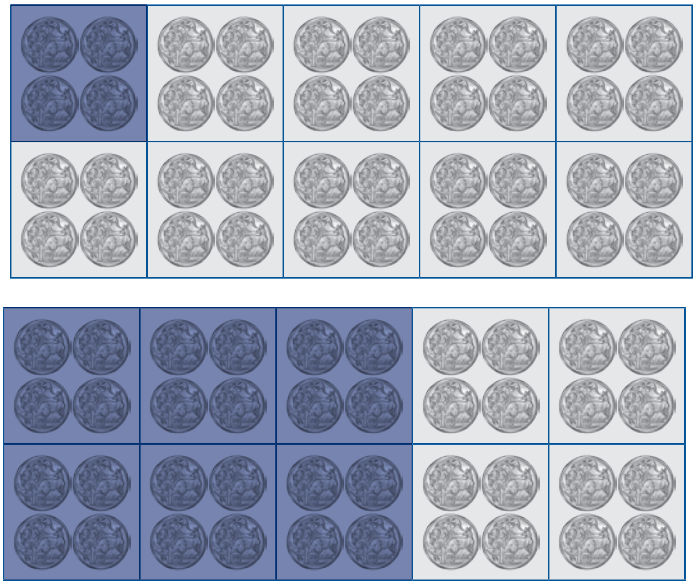

Figure 5. Finding six-tenths of $40.

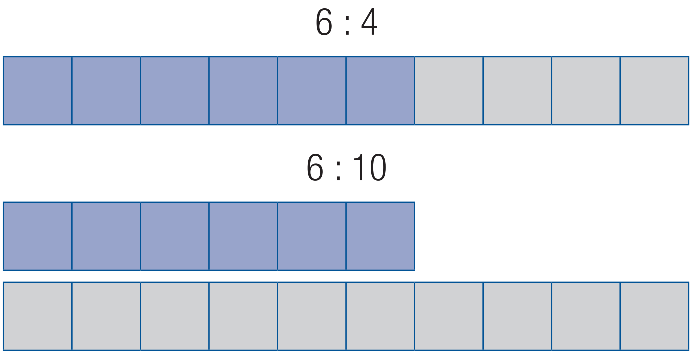

Figure 6. Part-whole and comparison ratios.

fraction prototype might express this as $90 \times 6 = 6 \times 90$ so it follows that $90 \times \frac{6}{10} = \frac{6}{10} \times 90$. You often split factors up when you multiply whole numbers. Most people find 8×23 by calculating 8×20 and 8×3, then adding the products. With your six-tenths prototype you can do the same. Six-tenths is five-tenths (one-half) plus one-tenth, so you could work out $\frac{6}{10} \times 90$ as $\frac{1}{2} \times 90 + \frac{1}{10} \times 90$. So a fraction prototype that is worthwhile can partition itself into other fractions when it is useful.

The ratio feature

Ratios are all around us: in the mixes for weedkiller, cement and paint; the gears on bicycles; and the facial dimensions of supermodels. It is not that our prototypical fraction turns into a ratio. Rather fractions are found in ratios in two main ways. Figure 6 demonstrates two different ratios in which six-tenths can be found. The ratios are different, so six-tenths will represent different relationships.

These different types are sometimes called part–whole and comparison ratios (Watanabe, 2002). In the first ratio, six-tenths represents the fraction of the whole (ten cubes) that is the blue part (six cubes). In the second ratio, six-tenths represents what fraction the blue quantity is of the yellow quantity. This is a subtle variation to the operator feature since six-tenths of ten is six ($\frac{6}{10} \times 10 = 6$). When you mix a fruit cocktail of six parts current juice to four parts apple juice you are using the part–whole feature. When you notice that the length of a supermodel's nose is about six-tenths of the length between the pupils of her eyes, you are using a comparison ratio.

As with measures, there are an infinite number of ratios that are six-tenths of blue for every four-tenths of yellow. These are created by repeatedly copying 6:4 or equally dividing 6:4. This is the only way that flavour, colour, density and other characteristics of the ratio stay the same. Here are a few examples:

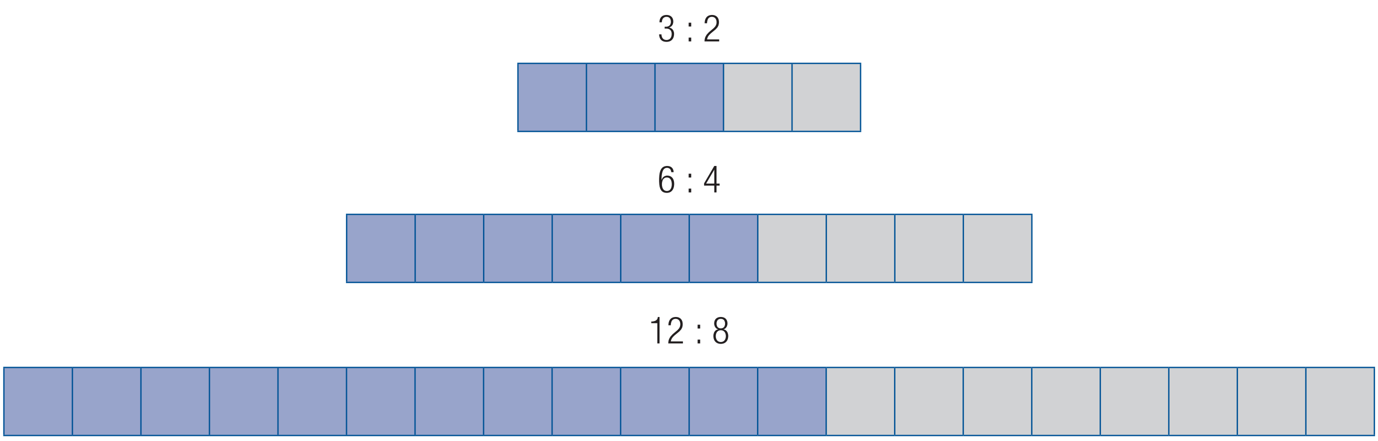

Figure 7. Equivalent ratios to 6:4.

A clever look at the prototype six-tenths in part–whole ratio form shows you the equivalent measure feature at work subtly under the surface. 3:2 is three-fifths blue, 6:4 is six-tenths blue and 12:8 is twelve-twentieths blue. Note that $\frac{3}{5}$, $\frac{6}{10}$ and $\frac{12}{20}$ are equivalent fractions, even though the whole is changing.

The quotient feature

Just when you thought your prototypical six-tenths had exhausted its features, it pulls out another very practical application. With the outdated whole-number model, sharing items can be a problem. Imagine sharing five doughnuts equally among two people. Each person gets two doughnuts but the remaining one doughnut has to be thrown away. It is all very wasteful. Of course with fractions the remaining one can be cut into halves so each person gets two and a half doughnuts. So what happens when six doughnuts are shared equally among ten people?

Start by imagining what happens when one doughnut is shared equally among ten people. It is obvious that each person will get one-tenth of the doughnut. That is hardly a mouthful.

Figure 8. Sharing one doughnut among ten people.

If there are six doughnuts and each doughnut is cut into tenths, then each person gets one-tenth of each doughnut. That is a total share for each person of six-tenths of one doughnut. You can write this as $6 \div 10 = \frac{6}{10}$, where division is being treated as sharing. Like most appealing features, the way it works is predictable. Imagine three doughnuts shared equally among five people. Each person gets three-fifths of a doughnut. This can be written as $3 \div 5 = \frac{3}{5}$. Imagine 12 doughnuts shared among 20 people: $12 \div 20 = \frac{12}{20}$ so each person gets twelve-twentieths of a doughnut. You might notice that $\frac{3}{5}$, $\frac{6}{10}$, and $\frac{12}{20}$ are equivalent fractions. If you look below, you may notice an interesting thing about the quotient feature: twice as many people sharing twice as many doughnuts results in the same sized share for each person.

$$3 \div 5 = \frac{3}{5}$$
$$6 \div 10 = \frac{6}{10}$$
$$12 \div 20 = \frac{12}{20}$$

Looking ahead

So prototypes are not a bad thing. The issue is not whether or not you have prototypes because humans need to think with prototypes. Prototypes allow you to compress key features of situations into a single artefact with which you can think. The important criterion is the sophistication of the prototype. Six-tenths, with all its interesting features, is a more useful prototype than one-half, seen only as a part of a pie. Just like the coal range was once a great prototype, our previously useful ideas give way to more sophisticated ones as the need arises. Who wants to use a coal range for cooking when a modern stove or microwave is available?

When next you teach fractional number, create a chart of your favourite prototypical fraction. Concept maps based on a single example like six-tenths that can be built on and modified are one excellent way to do this. You might do this as shared knowledge with the whole class or ask each student to

create their own chart as an indicator of his or her growing understanding. Encourage your students to change and add to their chart regularly, particularly as they encounter new contexts in which their fraction can be used.

For example, consider another situation to which six-tenths applies: chance. The probability feature is hard to control as you never quite know what you are going to get (Alatorre & Figueras, 2005). Just when you think six-tenths is under control, it goes into variability mode. What does six-tenths look like as a probability? Here is a spinner created using an online applet (Utah State University, 2012). There is a six-tenths chance of landing on blue when you spin it.

So what actually happens when you spin the spinner ten times? Six out of ten times it should land on blue, right? Well, yes and no. Here are some results:

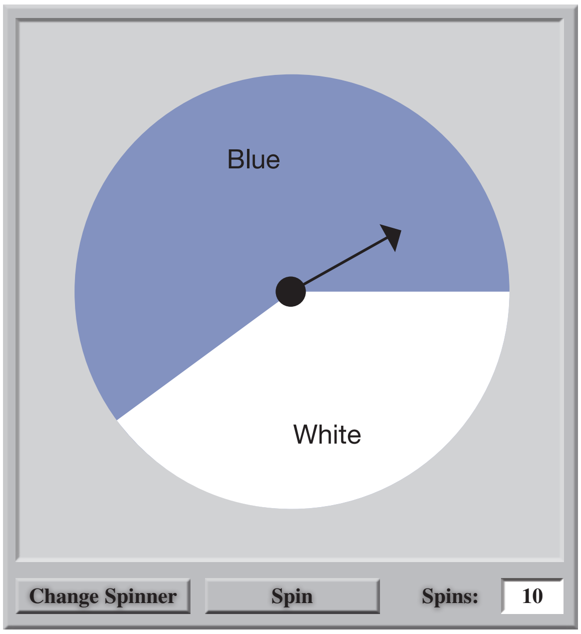

Figure 9. A spinner that is six-tenths blue.

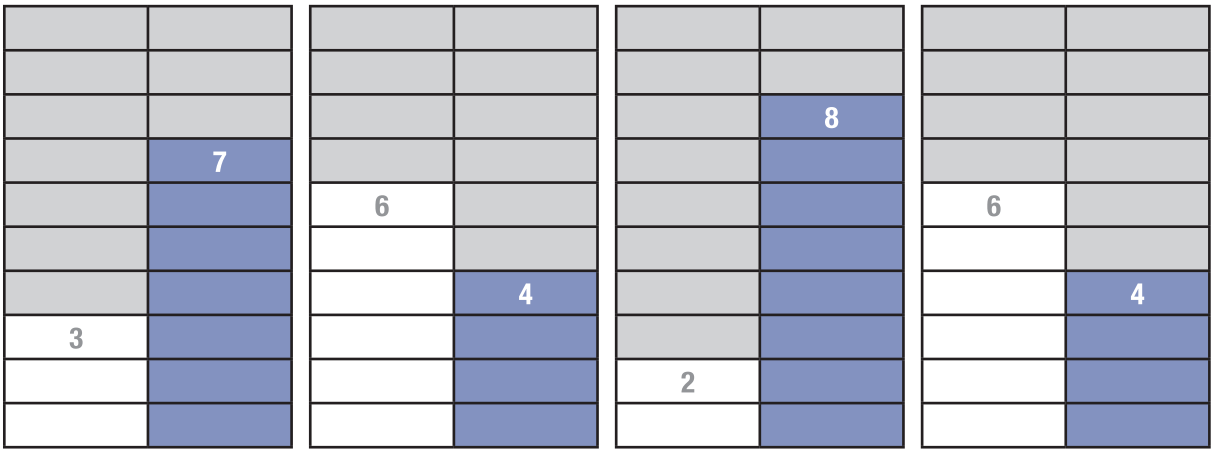

Figure 10. Four results of ten spins.

None of these results are six blue outcomes out of ten. If you keep generating samples of ten spins and graph the results on a dot plot you get a distribution something like this (Figure 11). Six-tenths is at the centre but there is a lot of variation.

So when they encounter new situations like this, students need to adapt the features of their prototypical fraction to suit. Managing their confidence is a delicate balance between having certainty and enabling flexibility. As a teacher, you should find ways for your students to reveal and share their prototypes. Let us get prototypes out of the private laboratory of individual student's minds and into the open shared space where they can inspire new innovations.

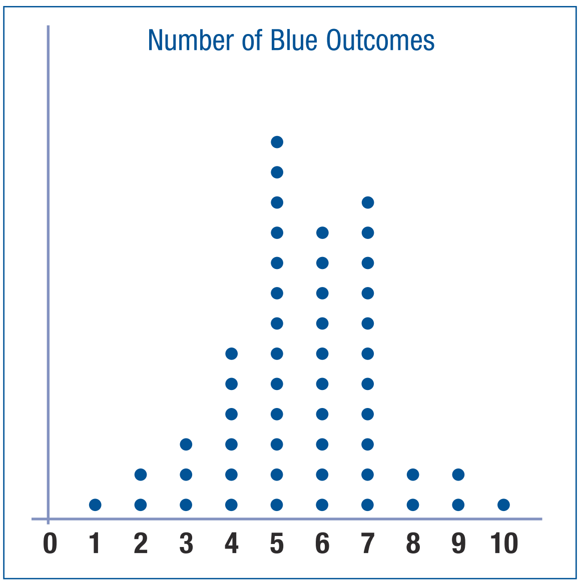

Figure 11. Dotplot of experiments of ten spins.

References

Alatorre, S. & Figueras, O. (2005). *A developmental model for proportional reasoning in ratio comparison tasks.* Paper presented at the 29th Annual Conference of the International Research Group for the Psychology of Mathematics Education, Melbourne, Australia.

Bransford, J. D., Brown, A. L. & Cocking, R. R. (2000). *How people learn: Brain, mind, experience and school.* Washington: National Research Council.

Fauconnier, G. & Turner, M. (2008). Rethinking metaphor. In R. Gibbs (Ed.), *Cambridge handbook of metaphor and thought.* New York: Cambridge University Press.

Gould, P. (2005). Really broken numbers. *Australian Primary Mathematics Classroom, 10*(3), 4–10.

Gray, E. & Tall, D. (2007). Abstraction as a natural process of mental compression. *Mathematical Education Research Journal, 19*(2), 23–40.

Hart, K. M., Kerslake, D., Brown, M. L., Ruddock, G., Kuchemann, D. E. & McCartney, M. (1981). *Children's understanding of mathematics: 11–16.* London: John Murray.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mancy, R. (2010). Rules, prototypes or examples? A framework for understanding transfer events. Retreived 1 June 2010 from http://www.rebeccamancy.net/Written/FrameworkForTransfer.pdf

Post, T., Behr, M., Lesh, R. & Wachsmuth, I. (1986). *Selected results from the rational number project.* Paper presented at the 9th Annual Conference of the International Group for the Psychology of Mathematics Education Antwerp, Netherlands.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds), *Rational numbers: An integration of research* (pp. 289–325). Hillsdale, NJ: Lawrence Erlbaum Associates.

Utah State University (2012). *National library of virtual manipulatives: Spinners.* Retrieved from http://nlvm.usu.edu/en/nav/frames_asid_186_g_3_t_5.html?open=activities&from=grade_g_3.html

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*(8), 457–463.

Yoshida, H. & Sawano, K. (2002). Overcoming cognitive obstacles in learning fractions: Equal partitioning and equal-whole. *Japanese Psychological Research, 44*(4), 183–195.